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On Approximations of Functions Preserving Symplectic Forms

SOBRE APROXIMAÇÕES DE FUNÇÕES PRESERVANDO FORMAS SIMPLÉTICAS

SOBRE APROXIMACIONES DE FUNCIONES QUE PRESERVAN FORMAS SIMPLÉCTICAS

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Resumo
O problema de aproximar um difeomorfismo (resp. fluxo) Ck preservador de volume ($k \geq 1$) em uma variedade compacta com ou sem fronteira por um difeomorfismo (resp. fluxo) foi originalmente motivado por considerações na teoria dos sistemas dinâmicos e proposto pela primeira vez por Palis e Pugh. Esse problema, apesar de sua aparente simplicidade para aqueles menos familiarizados com o assunto, esconde uma complexidade técnica e dificuldade extremamente sutis. O trabalho de Zehnder sobre técnicas de aproximação simplética oferece uma abordagem convincente para reexaminar os resultados fundamentais nessa área, conforme estabelecido por Palis e Pugh. Ao revisitar suas contribuições seminais por meio da perspectiva simplética de Zehnder, novas perspectivas podem surgir, avançando o estado da arte. Nesse contexto, revisaremos os resultados clássicos sobre aproximação e uma aproximação simplética, seguindo as ideias de Zehnder.

Palavras-chave: geometria simplética; aproximações de funções; sistemas dinâmicos.

Abstract
The problem of approximating a volume-preserving Ck diffeomorphism (resp. flow) ($k \geq 1$) on a compact manifold with or without boundary by a diffeomorphism (resp. flow) was originally motivated by considerations in dynamical systems theory and first posed by Palis and Pugh. This problem, despite its apparent simplicity for those less familiar with the subject , hides an extremely nuanced technical complexity and difficulty. Zehnder's work on symplectic approximation techniques provides a compelling avenue to re-examine the foundational results in this area as established by Palis and Pugh. Revisiting their seminal contributions through the lens of Zehnder's symplectic framework could yield novel insights and advance the state-of-the-art. With this in mind, we will revisit the classical results on approximation and a symplectic approximation following Zehnder's ideas.

Keywords: symplectic geometry; function approximations; dynamical systems.

Resumen
El problema de aproximar un difeomorfismo (resp. flujo) Ck preservador de volumen ($k \geq 1$) en una variedad compacta con o sin frontera mediante un difeomorfismo (resp. flujo) fue originalmente motivado por consideraciones en la teoría de sistemas dinámicos y planteado por primera vez por Palis y Pugh. Ese problema, a pesar de su aparente simplicidad para aquellos menos familiarizados con la materia, en realidad oculta una complejidad técnica y dificultad extremadamente sutil. El trabajo de Zehnder sobre técnicas de aproximación simpléctica ofrece una vía convincente para reexaminar los resultados fundamentales en esa área, tal como fueron establecidos por Palis y Pugh. Revisitar sus contribuciones seminales a través del marco simpléctico de Zehnder podría generar nuevas perspectivas y avanzar en el estado del arte. Con eso en mente, reexaminaremos los resultados clásicos sobre aproximación y una aproximación simpléctica siguiendo las ideas de Zehnder.

Palabras clave: geometría simpléctica; aproximaciones de funciones; sistemas dinámicos.

Introduction

Functions such as *sen*(*x*), e^x, *log*(*x*) belong to a class of functions called *analytic functions*, which means that around each point in their domain, there exists a power series representation of the form:

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Writing $f_n(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n$, we see that each f_n is a polynomial and $f(x)$= $\lim_{n\to\infty} f_n(x)$ for all x in the interval of convergence f_n of the series. It can also ben shown that within each compact interval of convergence, converges uniformly to f .

A generalization of the above result was proved by K. Weierstrass in 1885. According to Weierstrass, any continuous function $f:[a,b]\to\mathbb{R}$ can be uniformly approximated on its domain [*a, b*] by a sequence of polynomials. Specifically, given a continuous f on [*a, b*] and ϵ > 0, there exists a polynomial p such that $|f(x)-p(x)|<\epsilon$ for all $x\in[a,b]$. Note that we are approximating f by infinitely differentiable functions, since polynomials are C^∞. For a proof of Weierstrass' theorem, we recommend (Ransford, 1984).

In a more general context, we may ask whether a given continuous function f : $U\subseteq\mathbb{R}^n\to\mathbb{R}$,, where *U* is an open set, can be approximated by smooth functions. This is indeed possible, as the following example demonstrates. Define $\eta\in C^\infty(\mathbb{R}^n)$ by

$$\eta(x):=\begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1\end{cases}$$

where C is the is chosen $\int_{\mathbb{R}^r}\eta\,dx=1.$ Now, for each $\varepsilon>0$

$$\eta_\varepsilon(x):=\frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right).$$

It is straightforward to verify that the functions η_ε are smooth and satisfy $\int_{\mathbb{R}^n}\eta_\varepsilon\,dx=1$,

supp $\eta_\varepsilon\subseteq B(0;\varepsilon)$. Defining $f^\varepsilon(x):=\int_U\eta_\varepsilon(x-y)f(y)\,dy$, by a change of variables we can

write $f^\varepsilon(x)=\int_{B(0;\varepsilon)}\eta_\varepsilon(y)f(x-y)\,dy$. For reasons that will 1r be clear, f^ε is smooth and $f^\varepsilon\xrightarrow[\varepsilon\to0]{}f$ *f* uniformly on compact subsets of *U* .

In this paper, we focus on avolume-preserving approximations, which the above method does not always achieve. We will explore this issue through the lens of Symplectic Geometry.

We define (*M, σ*) a *symplectic manifold* where *M* is a smooth manifold and *σ* is a closed and non-degenerate 2-form. A diffeomorphism ϕ: $(M_1, \sigma_1)\to(M_2, \sigma_2)$.

A relationship between symplectic manifolds is described as follows: if $\phi^*\sigma_2=\sigma_1$ in which ϕ^* denotes the pullback in differential geometry. For example, given $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)\in\mathbb{R}^{2n}$

We can consider the 2-form

$$w_0(p,q) = \sum_{i=1}^{n} dp_i \wedge dq_i,$$

and with this form, $(\mathbb{R}^{2n}, w_0)$ is a symplectic manifold. Despite being a very simple example, it is always what occurs, in local coordinates, in any symplectic manifold. This is a classical result of symplectic geometry demonstrated by Darboux. We will give a proof of this fact following the ideas of (Moser, 1965) and (Zehnder, 1977).

An elementary reason for choosing symplectic geometry to try to solve the problem of approximating functions while preserving volume is that the symplectic diffeomorphisms preserve symplectic volumes and, therefore it suffices to smooth out these diffeomorphisms. More precisely, if $\phi : (M_1, \sigma_1) \to (M_2, \sigma_2)$ is a diffeomorphism between symplectic manifolds such that $\phi^*\sigma_2 = \sigma_1$ then $(\sigma_1)^n$ and $(\sigma_2)^n$ are volume forms on M_1 and M_2, respectively, and

$$\int_A (\sigma_1)^n = \int_{\phi(A)} (\sigma_2)^n,$$

for all Borel sets $A \subset M_1$.

This work is organized as follows. In section 2 we will discuss classical methods for function approximations. In section 3 we will revisit Zehnder's result following the original ideas (Zehnder, 1977). Finally, in section 4 we will see recent results about the problem of approximating a volume-preserving diffeomorphism.

Standard smoothness

Smoothness of functions is closely related to their density in specific functional spaces. For example, smooth functions are dense in the L^2 and L^1 spaces. This means that for any function whitin these spaces it is possible to approximate arbitrarily closely by smooth functions. One common approach to achieve this approximation is by convolving the function with a sequence of smooth functions with special properties. The resulting convolutions are smooth functions that converge to the original function in both the L^2 and L^1 norm. This result will be proved in detail in this section following the original ideas (Abraham; Marsden; Ratiu, 2012), which is already considered a classical result.

Theorem 2.1 (See Abraham; Marsden; Ratiu, 2012)). *Let M be a compact manifold. The subset* $C^\infty(M, \mathbb{R}^s)$ *is dense in* $C^p(M, \mathbb{R}^s)$, $p \geq 1$.

As established by (Abraham; Marsden; Ratiu, 2012), Theorem 2.1 demonstrates that smooth functions are dense in the space of p-integrable functions on a compact manifold M. This classical result forms the foundation for our study of structure-preserving function approximation using tools from symplectic geometry.

Unfortunately, method, does not guarantee that the approximating function preserves volume. However, all is not lost. In Section 3 we will explore how we can address this problem by using generating function techniques from the perspective of symplectic geometry.

Symplectic geometry offers a natural framework fo address this issue. As pioneered by Moser in his seminal work (Moser, 1965), the key idea is that symplectic diffeomorphisms preserve symplectic volumes. Therefore, if we can approximate our function using a symplectic isotopy, the volume will be preserved throughout the approximation.

In Section 3, we will examine Moser's path method for creating symplectic isotopies. By smoothly deforming a given function through a family of generating functions, we can achieve smooth approximations that precisely preserve the geometric structures encoded by the symplectic form. This approach elegantly avoids the issues that can occur when naively using mollification or convolution methods.

I believe that exploring these symplectic techniques will yield new insights into structure- preserving function approximation problems. It is exciting to continue advancing this area, which lies at the intersection of analysis, geometry and physics. I look forward to presenting our findings in Section 3 and potentially encouraging further progress on this topic.

Symplectic smoothing

In this section, we will improve the main result obtained in the previous section. In addition to approximating functions with infinitely differentiable functions, we will also ensure that the volume is preserved . With this objective, our primary result will be presented in the following theorem.

Theorem 3.1 – (See (ZEHNDER, 1977))**.** *Let (M, σ) and (N, τ) be symplectic manifolds. The set of smooth symplectic diffeomorphisms of class C^∞ from M to N is dense in the space of symplectic diffeomorphisms of class C^k from M to N, for $k \geq 1$.*

To present the text in an interesting and clear order, we will first demonstrate the following lemma:

Lemma 3.2 – *Let $W \subset \mathbb{R}^{2n}$ be open. Consider the set* $\mathcal{D}^{k+1}(W) := \{S \in C^{k+1}(W)$: *then* $(\frac{\partial^2 S}{\partial x \partial \eta}(x,\eta)) \neq 0, \forall (x,\eta) \in W\}$, $k \geq 1$. *If* $S, S_1 \in \mathcal{D}^{k+1}(W)$ *and* $\|S - S_1\|_{C^{k+1}(K_1)}$ *then* $\|E(S) - E(S_1)\|_{C^k(K_2)}$ *where $K_1 \subset W$ and K_2 contained in the domain of $E(S)$ are compact sets.*

To prove this lemma, we will need to state and prove the next two propositions in sequence.

Proposition 3.3 – *Let U, V $\subset \mathbb{R}^{2n}$ be open sets and $K1 \subset U$ and $K2 \subset V$ compact sets. Denote by $Dif^k(U,V)$ the set of class C^k diffeomorphisms from U to V . For $k \geq 1$, if f, g $\in Dif^k(U, V)$ and $\| f - g\|_{Ck(K1)}$ then $\|f^{-1} - g^{-1}\|_{Ck(K2)}$.*

Proof. We will prove this by induction on k. Assume that f, g : $K_1 \to K_2$ are still diffeomorphisms. Begin with $k = 1$. Given $\epsilon > 0$, suppose that $\|f - g^{-1}\|_{C1(K1)} < \frac{\epsilon}{2}$, i.e. $|f(x) - g(x)| < \frac{\epsilon}{2}$ and $|Df(x) - Dg(x)| < \frac{\epsilon}{2}$ for all $x \in K_1$. Since $f|_{K_1}$ is uniformly continuous, given $x, y \in K_1$ with $|x - y| < \frac{\epsilon}{4}$, we can assume that

$$|f(x) - g(y)| \leq |f(x) - f(y)| + |f(y) - g(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

An analogous idea applies to the functions Df and Dg. Since f is a diffeomorphism, given an arbitrary $w \in K_2$ there exists $a_1 \in K_1$ such that $f(a_1) = w$. By the above, $f(a_1) \in g(B(a_1; \frac{\epsilon}{4}) \cap K_1) \subset B(w; \epsilon)$. Thus, there exists $a_2 \in B(a_1; \frac{\epsilon}{4}) \cap K_1$ such that $g(a_2) = w$. It follows that

$$|f^{-1}(w) - g^{-1}(w)| = |f^{-1}(f(a_1)) - g^{-1}(g(a_2))| = |a_1 - a_2| < \frac{\epsilon}{4} < \epsilon.$$

Since w is arbitrary, $|f^{-1}(w) - g^{-1}(w)| < \epsilon$ for all $w \in K_2$.

Noting that $f^{-1} \circ f = id$ and $g^{-1} \circ g = id$, we have $Df^{-1}(f(x)) = [Df(x)]^{-1}$ and $Dg^{-1}(g(x)) = [Dg(x)]^{-1}$ for all $x \in K_1$. Since the inversion of matrices with non-zero determinant is continuous, and $|Df(x) - Dg(y)| < \frac{\epsilon}{2}$, we can assume $[Df(x)] - [Dg(y)]^{-1} < \epsilon$ for all $x, y \in K_1$. On the other hand, for each $w \in K_2$ there exist $a_1, a_2 \in K_1$ with $w = f(a_1) = g(a_2)$ and since $[Df(a_1)]^{-1} - [Dg(a_2)]^{-1} < \epsilon$. Therefore, we have proven that $||f^{-1} - g^{-1}||_{C^1(K_2)} < \epsilon$.

Now, assume that the proposition holds true for $k > 1$. Given $\epsilon > 0$, suppose that $||f - g||_{C^{k+1}(k_1)} < \frac{\epsilon}{2}$. Note that, for any point $x \in K$, $Df(x)$ and $Dg(x)$ are C^k diffeomorphisms. Since $||Df(x) - Dg(x)||_{C^k} < \frac{\epsilon}{2}$, for all $x \in K_1$, by the induction hypothesis, we can assume that $||[Df(x)]^{-1} - [Dg(x)]^{-1}||_{C^k} < \epsilon$. Since $Df^{-1}(f(x)) = [Df(x)]^{-1}$ and $Dg^{-1}(g(x)) = [Dg(x)]^{-1}$ for every $x \in K_1$, we have $||Df^{-1}(f(x)) - Dg^{-1}(g(x))||_{C^k} < \epsilon$. In particular, since f and g are diffeomorphisms, $||Df^{-1}(y) - Dg^{-1}(y)||_{C^k} < \epsilon$, for all $y \in K_2$. As we have seen in previous cases, we can assume that $|f^{-1}(y) - g^{-1}(y)| < \epsilon$ for all $y \in K_2$. Thus, $||f^{-1} - g^{-1}||_{C^{k+1}(K_2)} < \epsilon$.

Proposition 3.4 – *Let $U, V, W \subset \mathbb{R}^{2n}$ be open sets. Then, for $k \geq 1$,*

a. *Let $\Phi : V \to W$ be a C^k function, $K_1 \subset U$ and $K_2 \subset U$ and k_2 be compact sets, and $h_1, h_2 \in C^k(U, V)$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $||h_1 - h_2||_{C^k(K_1)} < \delta$, then $||\Phi \circ h_1 - \Phi \circ h_2||_{C^k(K_1)} < \epsilon$.*

b. *Let $\Psi : U \to V$ be a C^k function, $K \subset U$ be a compact set, and $f_1, f_2 \in C^k(V, W)$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $||f_1 - f_2||_{C^k(\Psi(K))} < \delta$, then $||f_1 \circ \Psi - f_2 \circ \Psi||_{C^k(K)} < \epsilon$.*

c. *Let $f_1, f_2 \in C^k(U, V)$ and $g_1, g_2 \in C^k(V, W)$. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $||f_1 - f_2||_{C^k(K_1)} < \delta$ and $||g_1 - g_2||_{C^k(K_2)} < \delta$, then $||g_1 \circ f_1 - g_2 \circ f_2||_{C^k(K_1)} < \epsilon$, where $K_1 \subset U$ and $K_2 \subset V$ are compact sets.*

Proof. We will individually present the demonstration for each item.

a. Given $\epsilon > 0$, suppose that $||h_1 - h_2||_{C^k(k1)} < \frac{\epsilon}{||\Phi||_{C^k(K_2)}}$, i.e. $|h_1(x) - h_2(x)| < \frac{\epsilon}{||\Phi||_{C^k(K_2)}}$ *and* $|D^j h_1(x) - D^j h_2(x)| < \frac{\epsilon}{||\Phi||_{C^k(K_2)}}$ is uniformly continuous, we can assume that $|\Phi \circ h_1(x) - \Phi \circ h_2(x)| < \epsilon$ for all $x \in K_1$. Moreover, for all $x \in K_1$ and $j = 1, \ldots k$, $|D^j(\Phi \circ (h_1(x) - h_2(x)))| \leq |D^j\Phi(h_1(x) - h_2(x))|$

$$\begin{aligned} &\leq |D^j\Phi(h_1(x) - h_2(x))|\,|D^j(h_1(x) - h_2(x))| \\ &\leq ||\Phi||_{C^k(K_2)} ||h_1 - h_2||_{C^k(K_1)} < \epsilon, \end{aligned}$$

Therefore, $\|\Phi \circ h_1 - \Phi \circ h_2\|_{C^k(K_1)} < \epsilon$.

b. Note that

$$\begin{aligned} |D^j(f_1(\Psi(x)) - f_2(\Psi(x)))| &\leq |D^j f_1(\Psi(x)) - D^j f_2(\Psi(x))|\,|D^j\Psi(x)| \\ &\leq \|f_1 - f_2\|_{C^k(\Psi(K))}\|\Psi\|_{C^k(K)} \end{aligned} \tag{1}$$

for all $x \in K$ and $j = 1, \ldots k$. We will verify the two possible cases. ***Case 1.*** $\|\Psi\|_{C^k(K)} \leq 1$. In this case, suppose that $\|f_1 - f_2\|_{C^k(\Psi(K))} < \epsilon$. In particular, $|f_1(\Psi(x)) - f2(\Psi(x))| < \epsilon$ for all $x \in K$. Moreover, by (1), $|D^j(f_1(\Psi(x)) - f_2(\Psi(x)))| < \epsilon$, for all $x \in K$ and $j = 1, \ldots k$. Therefore, $\|f_1 \circ \Psi - f2 \circ \Psi\|_{C^k(K)} < \epsilon$.
Case 2. $\|\Psi\|_{C^k(K)} > 1$. For this case, suppose that $\|f_1 - f_2\|_{C^k(\Psi(K))} < \frac{\epsilon}{\|\Psi\|_{C^k(K)}}$. This implies that $|f_1(\Psi(x)) - f_2(\Psi(x))| < \frac{\epsilon}{\|\Psi\|_{C^k(K)}}$ for all $x \in K$. . By (1), $|D^j(f_1(\Psi(x)) - f_2(\Psi(x)))| < \epsilon$, for all $x \in K$ and $j = 1, \ldots k$. Since $\frac{\epsilon}{\|\Psi\|_{C^k(K)}} < \epsilon$, $|f_1(\Psi(x)) - f_2(\Psi(x))| < \epsilon$ for all $x \in K$. Therefore $\|f_1 \circ \Psi - f_2 \circ \Psi\|_{C^k(K)} < \epsilon$.

c. Let $K_1 \subset U$ and $K_2 \subset V$ be compact sets and $\epsilon > 0$. By item (a), there exists $\delta_1 > 0$, such that if $\|f_1 - f_2\|_{C^k(K1)} < \delta_1$ then $\|g_2 \circ f_1 - g_2 \circ f_2\|_{C^k(K1)} < \frac{\epsilon}{2}$. By item ($b$), there exists $\delta_2 > 0$, such that if $\|g_1 - g_2\|_{C^k(K_2)} < \delta_2$ then $\|g_1 \circ f_1 - g_2 \circ f_1\|_{C^k(K_1)} < \frac{\epsilon}{2}$. Therefore, given $\epsilon > 0$, take $\delta = \max\{\delta_1, \delta_2\}$, and if $\|f_1 - f_2\|_{C^k(K_1)} < \delta$ and $\|g_1 - g_2\|_{C^k(K_2)} < \delta$ it implies that $\|g_1 \circ f_1 - g_2 \circ f_2\|_{C^k(K_1)} \leq \|g_1 \circ f_1 - g_2 \circ f_1\|_{C^k(K_1)} + \|g_2 \circ f_1 - g_2 \circ f_2\|_{C^k(K_1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

In light of these results, we can easily proceed with the proof of the lemma that was left pending at the beginning of this section.

Proof of Lemma 3.2 – *Let* $S_1, S_2 \in D^{k+1}(W)$ and $K \subset W$ be compact. Given $\epsilon > 0$, choose $\delta > 0$ such that $\|S_1 - S_2\|C^k{+}1(K) < \delta$. Take U as the codomain of the mapping $\sigma_1 : W \to U$ defined by

$$\sigma_1(x, \eta) = \left(x, \frac{\partial S_1}{\partial x}(x, \eta)\right).$$

We know that both the mapping σ_1 and $\sigma_2 : W \to V$ defined by $\sigma_2(x, \eta) = \left(\frac{\partial S_1}{\partial \eta}(x, \eta), \eta\right)$ are diffeomorphisms (see (Abraham; Marsden; Ratiu, 2012)). Similarly, for the mappings $\tau_1 : W \to U$ and $\tau_2 : W \to V$ defined by

$$\tau_1(x, \eta) = \left(x, \frac{\partial S_2}{\partial x}(x, \eta)\right)$$
$$\tau_2(x, \eta) = \left(\frac{\partial S_2}{\partial \eta}(x, \eta), \eta\right).$$

Clearly, since S_1 and S_2 are C^k-close, $\|\sigma_1 - \tau_1\|_{C^k(K)} < \delta$ and $\|\sigma_2 - \tau_2\|_{C^k(K)} < \delta$. Hence, by Proposition 3.3, the functions σ_1^{-1} and τ_1^{-1} are C^k-close. Let $F_1 := E(S_1)$ and $F_2 := E(S_2)$. By (Abraham; Marsden; Ratiu, 2012), we can rewrite these functions as:

$$\begin{aligned} F_1 &= \sigma_2 \circ \sigma_1^{-1} \\ F_2 &= \tau_2 \circ \tau_1^{-1}. \end{aligned}$$

Therefore, by item (*c*) of Proposition 3.4, $\|F_1 - F_2\|_{C^k(K_1)} < \epsilon$, where $K_1 \subset U$ is compact.

Note tha we can smooth out a symplectic diffeomorphism $F = E(S)$ through a standard smoothing of its generating function S. Indeed, let $F = E(S)$ be a C^k-symplectic diffeomorphism, $k \geq 1$, from $U \subset \mathbb{R}^n$ to $V \subset \mathbb{R}^n$, where U is simply connected and $\bar{U}$ is compact, which is given by the generating function $S \in C^{k+1}(W)$ (See (Abraham; Marsden; Ratiu, 2012). We can write $F = \sigma_2 \circ \sigma_1^{-1}$ where $\sigma_1 : W \to U$ and $\sigma_2 : W \to V$ are diffeomorphisms. We will choose open subsets W_2, $W_3 \subset W$ such that $\overline{W}_3$, $\overline{W}_2$ are compact and $\overline{W}_3 \subset \overline{W}_2$ and $\overline{W}_2 \subset W$ (we abbreviate this as $W_3 \Subset W_2 \Subset W$). To approximate S on W_3 by a C^∞ function, begin by choosing functions $\zeta \in C^\infty(W_2)$ with $\zeta \equiv 1$ on W_3 and $\gamma \in C^\infty(W)$ with $\gamma \equiv 1$ on W_2. Given $\epsilon > 0$, there exists a function $X_\epsilon \in C^\infty(W)$ such that

$$\|\gamma \cdot [(\zeta \cdot S) - X_\epsilon * (\zeta \cdot S)]\|_{C^{k+1}(W)} < \epsilon.$$

Define $S_1 \in C^{k+1}(W)$ by

$$S_1 = S - \gamma \cdot [(\zeta \cdot S) - X_\epsilon * (\zeta \cdot S)].$$

It follows that $\|S - S_1\|_{C^{k+1}(W)} < \epsilon$. Furthermore, $S_1 = S$ on $W \setminus W_2$, since $\zeta \equiv 0$ outside W_2, and $S_1 = X_\epsilon * S$ on W_3, since $\gamma \equiv \zeta \equiv 1$ on W_3. Therefore $S_1|_{W_3} \in C^\infty$. Now, for ϵ sufficiently small we choose $U_3 \Subset U_2 \Subset U$ such that $U_3 \subset \sigma_1(W_3)$ and $\sigma_1(W_2) \subset U_2$. Define

$$F_1 := E(S_1).$$

We can write $F_1 = \tau \circ \tau^{-1}$, where $\tau(x, \eta) = (x, \frac{\partial S_1}{\partial x}(x, \eta))$ and $\tau(x, \eta) = (\frac{\partial S_1}{\partial \eta}(x, \eta), \eta)$, is sufficiently small. In this case, the following properties hold:

(P1) $F_1|_U \in C^\infty(U_3)$, since τ_1 and τ_2 are C^∞ on W_3.

(P2) As we can see in (Abraham; Marsden; Ratiu, 2012), F_1 is a symplectic diffeomorphism from U to $F(U)$.

(P3) $F_1 = F$ in $U \setminus U_2$, since $S = S_1$ in $W \setminus W_2$, which implies $\sigma_1 = \tau_1$ and $\sigma_2 = \tau_2$ in $W \setminus W_2$.

(P4) F_1 is of class C^p, $k \leq p \leq \infty$, in the open sets where F is of class C^p, because in these open sets the function S_1 defined above will be of class C^{p+1}, and since $F_1 = E(S_1)$, F_1 will be of class C^p.

(P5) By Lemma 3.2, $\|F_1 - F\|_{C^k(U)} < \delta(\epsilon)$, where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.

Now, we can proceed with the approximation while preserving the volume as announced at the beginning of the section.

Proof of Theorem 3.1 – Let $f \in C^k(M, N)$ be a symplectic diffeomorphism from M to N and Wf be a sufficiently small open neighborhood of f in the C^k topology. Choose a locally finite covering of M, consisting of symplectic charts $(Ui, \phi i)$, $1 \leq i \leq \infty$, with the following properties:

b. $\overline{U_i}$ is compact;

b. $h(U_i) \subset V_i$ for all $h \in C^k(M, N) \cap W_f$, with (V_i, ϕ_i), $1 \leq i \leq \infty$ being an atlas in N.

Additionally, take a covering $(U^{(3)})$, $1 \leq i \leq \infty$, of M, such that $U_i^{(3)} \Subset U_i^{(2)} \Subset U_i$ and in U_i every symplectomorphism in the neighborhood W_f is given by the local generating function of f as in (Abraham; Marsden; Ratiu, 2012), where the local construction is previously taken with respect to $\phi_i(U^{(3)}) \Subset \phi_i(U^{(2)}) \Subset \phi_i(U_i)$. Now, we will define a sequence of functions (f_n) such that $f_0 = f$ and $f_n|_{U_1^{(3)} \cup \cdots \cup U_n^{(3)}}$ is C^∞.

Based on properties of our cover, the local map $F_1 := \phi_1 \circ f \circ \phi^{-1} \in C^k(\phi_1(U_1), \phi_1(V_1))$ is a symplectomorphism from $\phi_1(U_1)$ to its image, which is given by $F_1 := E(S_1)$, where S_1 is the local generating function of F_1. As we saw earlier, there exists a map G_1 such that

- $G_1 = F_1$ outside of ϕ_1('$U^{(2)}$);
- $G_1|_{\phi_1(U^{(3)})}$ is of class C^∞;
- G_1 is C^p in the open sets where F_1 is C^p;
- G_1 is a symplectomorphism;
- There exists $\delta_1 > 0$ such that $\|G_1 - F_1\|_{C^k(\phi_1(U_1))} < \delta_1$.

Let $f_0 = f$. Define the function f_1 as follows:

$$f_1(z) = \begin{cases} f_0(z), & z \in M \setminus \overline{U_1^{(2)}} \\ \left(\varphi_1^{-1} \circ G_1 \circ \phi_1\right)(z), & z \in U_1 \end{cases}$$

Note that $f_1|U_1^{(3)}$ is C^∞ and is of class C^p in the open sets where f_0 is C^p. Moreover, if δ_1 is chosen sufficiently small, then $f_1 \in C^k(M, N) \cap W_f$ and is a symplectomorphism.

Assume that, for $n \geq 2$, $f_{n-1} \in C^k(M, N) \cap W_f$ is defined and satisfies the following:

a. f_{n-1} is a symplectomorphism;
b. $f_{n-1}|_{U(3)1 \cup \cdots \cup U(3)n-1}$ is C^∞;
c. f_{n-1} is Cp in the open sets where f_{n-2} is C^p.

Therefore, for the symplectomorphism $F_n := \phi_n \circ f_{n-1} \circ \phi^{-1} \in C^k(\phi_n(U_n), \phi_n(V_n))$, there exists a function Gn such that

- $G_n = F_n$ outside of $\phi_n(U^{(2)}_n)$;
- $G_n|_\phi (U^{(3)})$ is of class C^∞;
- G_n is C^p in the open sets where F_n is C^p;
- G_n is a symplectomorphism;
- There exists $\delta_n > 0$ such that $\|G_n - F_n\|_{C^k} (\phi_n(U_n)) < \delta_n$.

Thus, define

$$f_n(z) = \begin{cases} f_{n-1}(z), & z \in M \setminus \overline{U_n^{(2)}} \\ \left(\varphi_n^{-1} \circ G_n \circ \phi_n\right)(z), & z \in U_n \end{cases}$$

If δ_n is chosen sufficiently small, then f_n is a symplectomorphism from M to N, which belongs to the neighborhood W_f of f, and $f_n|_{U_1^{(3)} \cup \ldots \cup U_{n-1}^{(3)} \cup U_n^{(3)}}$ is of class C^∞.

Finally, define $g(z) := \lim_{n \to \infty} f_n(z)$. This limit is not difficult to compute since $n \to \infty$ the cover $(U^{(3)})$ is locally finite. With the appropriate choice of (δ_n), we conclude that $g \in C^\infty(M, N) \cap W_f$ is a symplectomorphism from M to N and satisfies the desired properties.

Approximations C1 and other results

The question of approximating a volume-preserving C^k ($k \geq 1$) diffeomorphism (or flow) on a compact manifold with or without boundary via a diffeomorphism (or flow) stems from dynamical systems theory and was proposed by Palis and Pugh (cf. (Palis; Pugh, 1975). This issue, notwithstanding its facial simplicity for those less acquainted with the material, conceals an intricacy and finely-honed technical challenge.

So far, answers to this problem are partial, and the main question, when $k = 1$, remains open. Interest in studying this issue has solidified recently with the development of techniques in dynamical systems that generically apply to C^1 diffeomorphisms, and with the development of techniques in ergodic theory that apply to C^2 diffeomorphisms. A positive answer to such a question would lead us to interesting connections between dynamical systems and ergodic theory (see Arbieto-Matheus; Arbieto; Matheus, 2007), for an example of such connections). More recently, in (Avila; Crovisier; Wilkinson, 2021), the authors established a C^1 version of the stable ergodicity conjec- ture for partially hyperbolic volume-preserving diffeomorphisms. Their principal findings surmounted the limitations of the Pugh-Shub approach by introducing novel perturbation implements within the C^1 topology. These tools encompass the linearization of horseshoes and the generation of "superblenders" from hyperbolic sets exhibiting high entropy. It was proven that stable ergodicity is generic among nonuniformly Anosov diffeomorphisms. Furthermore, open questions and connections to other dynamical properties were discussed.

Therefore, in this section, we will present some partial results involving approximations of volume-preserving functions. Earlier, we discussed how Zehnder proved the case of symplectic diffeomorphisms on boundaryless surfaces. It is natural to inquire about potential extensions of such results. Let us now explore some of these extensions:

$C1$, α Approximations

We will denote by $C^{k,\alpha}$, $k \leq 0$ and $0 < \alpha < 1$, the usual Hölder space and if$f \in C^{k,\alpha}$ we define $\|f\|$ $\|f\|_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$.

For $C^{1,\alpha}$ diffeomorphisms preserving volume we have a satisfactory answer given by Zehnder, namely

Theorem 4.1 – (Zehnder, 1977)). *Let M be a compact* C$^\infty$ *manifold of dimension d with a volume form μ. Let f* $\in C^{1,\alpha}(M)$ *be a volume-preserving diffeomorphism,* $0 < \alpha \leq 1$. *Then f can be approximated by a* C$^\infty$ *volume-preserving diffeomorphism in the following sense. There exists a sequence* (f_n) *of* C$^\infty$ *diffeomorphisms with* $f^*\mu = \mu$, *such* that

$$\|f_n - f\|_{C^1} \xrightarrow{n\to\infty} 0 \tag{2}$$

$$\|f_n\|_\alpha \leq k, \tag{3}$$

with the constant $k > 0$ *depending on* $\|f\|_\alpha$, *but independent of n.*It is worth emphasizing that the proof of the above result uses Hodge Theory and completely differs from the symplectic case.

C1 Flows

Let M be an m-dimensional surface, $m \geq 2$, without boundary. We say that a vector field X is conservative if $div\, X = 0$ and we denote by $X \in \mathfrak{X}_m(M)$. An equivalent condition for a C^1 map f to preserve volume is that $|\det Df| \equiv 1$. A proof of this fact can be seen in (Viana; Oliveira, 2019).

Now, suppose that f^t is the flow associated to the C^1 vector field X. Liouville's formula expresses the Jacobian of f^t in terms of the divergence $div\, X$ of the vector field X:

$$\det Df^t(x) = \exp\left(\int_0^1 div\, X(f^s(x))ds\right) \quad (4)$$

We easily see that, by (4), if X is a conservative field then its flow preserves volume.

The result below proved by Zuppa, (Zuppa, 1979), (see also Arbieto-Matheus, (Arbieto; Matheus, 2007)), shows that C^1 flows can be approximated (in the sense of the C^1 topology) by C^∞ flows.

Theorem 4.2 (Zuppa, 1979)). $\mathfrak{X}^\infty(M)$ *is C^1-dense in* $\mathfrak{X}^1(M)$.

Approximations in Regions with Boundary

Regarding the issue of approximating diffeomorphisms in regions with boundaries of $\mathbb{R}^n$, we will see a stronger result of Moser's Theorem made by Dacorogna and Moser.

Consider $\Omega \subset \mathbb{R}^n$ an open, connected and bounded set and two volume forms τ, β

$$\tau = f(x)dx_1 \wedge \ldots dx_n, \ \beta = g(x)dx_1 \wedge \ldots dx_n,$$

with $f, g > 0$.

We can show, under certain regularity conditions on Ω, f, g, that there exists a diffeomorphism $\phi : \overline{\Omega} \to \overline{\Omega}$ keeping the boundary condition fixed and such that

$$\phi^*\beta = \lambda\tau$$

where $\lambda = \int \beta \,/ \int \tau.$

The result above is equivalent to

Theorem 4.3 – (Dacorogna; Moser, 1990)). *Let $k \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected set and with boundary $\partial\Omega$ of class $C^{k+3,\alpha}$. Let $f, g \in C^{k,\alpha}(\overline{\Omega})$, $f, g > 0$ in $\overline{\Omega}$. Then there exists a diffeomorphism ϕ with $\phi, \phi^{-1} \in C^{k+1,\alpha}(\overline{\Omega}; \mathbb{R}^n)$ satisfying*

$$\begin{cases} g(\varphi(x)) \det \nabla\varphi(x) = \lambda f(x), & x \in \Omega \\ \varphi(x) = x, & x \in \partial\Omega \end{cases} \quad (5)$$

For the proof of this theorem, see (DACOROGNA; MOSER, 1990).

It is interesting to note that the solution uses classical theory of Elliptic Partial Differential Equations, reducing the problem to finding a field that satisfies a problem of the type

$$\begin{cases} \mathrm{div} Y(x) = h(x), & x \in \Omega \\ Y(x) = 0, & x \in \partial\omega \end{cases} \quad (6)$$

where h is a suitably chosen C^α function.

A natural attempt to solve such a problem is to try to find a solution of the type div$Y = \nabla u$, which transforms equation (6) into the equation involving the Laplacian operator

$$\begin{cases} \Delta u = h(x), & x \in \omega \\ \nabla u(x) = 0, & x \in \partial\Omega \end{cases} \tag{7}$$

With traditional methods (Schauder estimates), the existence and regularity of solutions to equation (7) is proven. When trying to treat the case $\alpha = 0$ with these methods, we encounter equation (6) with the function h only continuous. In this case, we encounter the following negative result due to McMullen:

Theorem 4.4 (Mcmullen, 1998))**.** *For any $n > 1$ there exists a function $f \in L^\infty(\mathbb{R}^n)$* which is not the divergence of any Lipschitz vector field.

See the proof of this theorem in (Mcmullen, 1998). Based on this theorem, Bourgain and Brezis in (Bourgain; Brezis, 2002) obtained very interesting negative results in the study of equation 6, when h is only a continuous function.

Final considerations

This article aimed to provide a review of the main results surrounding the conjecture proposed in 1975 by Palis and Pugh (see Palis; Pugh, 1975)), including the detailed proof by Zehnder (see (Zehnder, 1977)). A section was dedicated to reporting the partial results since then, leading up to the recent 2021 result by Avila *et al.* (see Avila; Crovisier; Wilkinson, 2021), where they established a C^1 version of the stable ergodicity conjecture for partially hyperbolic volume-preserving diffeomorphisms. The authors' principal findings have overcome the limitations of the Pugh-Shub approach by introducing innovative perturbation techniques within the C^1 topology. These tools encompass the linearization of horseshoes and the generation of "superblenders" from hyperbolic sets exhibiting high entropy. It has been proven that stable ergodicity is a prevalent property among non-uniformly Anosov diffeomorphisms. Furthermore, the article discussed open questions and explored connections to other dynamical properties.

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