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The relativistic velocity of an electron submitted to an electric field

VELOCIDADE RELATIVÍSTICA DE UM ELÉTRON SUBMETIDO A CAMPO ELÉTRICO

VELOCIDAD RELATIVISTA DE UN ELECTRÓN SOMETIDO A UN CAMPO ELÉCTRICO

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Abstract

This study investigates the velocity of an electron subjected to a uniform and constant electric field, while accounting for a resistive force proportional to the electron's speed. To characterize the movement, Newton's equations from classical mechanics and the relativistic equations from relativistic mechanics were used. Using classical mechanics it is possible to obtain an analytical expression for velocity; however, this approach is not valid for high velocities. On the other hand, while relativistic mechanics is suitable for high velocities, it does not yield an analytical solution for speed. Consequently, in this case, the solution was obtained through numerical computation.

Keywords: relativistic velocity; velocity parameter; relativistic mechanics.

Resumo

Neste trabalho, determinamos a velocidade de um elétron submetido a um campo elétrico uniforme e constante, considerando uma força de resistência ao movimento proporcional à velocidade. Para caracterizar o movimento, foram utilizadas as equações de Newton da mecânica clássica e as equações da mecânica relativística. Desse modo, a utilização da mecânica clássica possibilita obter uma expressão analítica para a velocidade, embora esta não seja válida para altas velocidades, enquanto a mecânica relativística, aplicável a altas velocidades, não permite obter uma expressão analítica para a velocidade. Nesse caso, a solução foi obtida numericamente de forma computacional.

Palavras-chave: velocidade relativística; parâmetro de velocidade; mecânica relativística.

Resumen

En este trabajo, la velocidad de un electrón sometido a un campo eléctrico uniforme y constante se determina considerando una fuerza de resistencia al movimiento proporcional a la velocidad. Para caracterizar el movimiento se utilizaron las ecuaciones de Newton de la mecánica clásica y las ecuaciones relativistas de la mecánica relativista. Utilizando la mecánica clásica es posible obtener una expresión analítica para la velocidad, pero no es válida para altas velocidades; utilizando mecánica relativista válida para altas velocidades, no es posible obtener una expresión analítica para la velocidad, y en ese caso la solución se obtuvo numéricamente de forma computacional.

Palabras clave: velocidad relativista; parámetro de velocidad; mecánica relativista.

Introduction

In 1904, Hendrik Antoon Lorentz (1853-1928) discovered a transformation that preserved the form of Maxwell's equations, provided that the components of the field underwent suitable modifications (Resnick, 1968).

Although the Lorentz transformation provided a basis for the development of special relativity, the profound consequences of relativity were not discovered by Lorentz, who, at that time, still believed in the ether hypothesis and sought to reconcile his transformation into the ether framework of electromagnetism. The development of special relativity, as we understand it today, was ultimately advanced by Jules Henri Poincaré (1854-1912) and Albert Einstein (1879-1955) (Resnick, 1968).

In early 1899, Poincaré proposed that the laws of nature should remain consistent for two observers moving uniformly relative to one another. He called this the "Principle of Relativity". Poincaré also concluded that a new type of dynamics would need to be developed, one which would, among other characteristics, adhere to the principle that no speed could surpass the speed of light (Blokhintsev, 1966).

In 1905, Einstein published his article titled: "Electrodynamics of Movies Bodies" (Einstein, 1905), in which he developed the special theory of relativity based on two basic postulates: (i) the principle of relativity and (ii)the constancy of the speed of light. Einstein derived the transformations of various physical quantities when shifting from one frame of reference to another and demonstrated how Newton's laws of classical mechanics needed to be modified accordingly.

In this study we analyze the movement of an electron exposed to a constant and uniform electric field, simultaneously with the presence of a resistive force that is proportional to its velocity, considering the classical and relativistic cases.

The physical situation of the problem

At the moment, $t_0 = 0$, a constant (does not depend on time) and uniform (does not depend on position) electric field $\vec{E}$ is applied to a stationary particle with charge q and mass m at rest, $\vec{v}_0 = 0$. Consequently, the particle experiences an electrical force as described by (Vanderlinde, 2004)

$$\vec{F}_e = q\vec{E}. \tag{1}$$

In addition to the force , we consider the existence of a resistance force $\vec{f}$ that opposes the movement of the particle and is proportional to the velocity, that is

$$\vec{f} = -\alpha\vec{v}, \tag{2}$$

where α is a constant of proportionality. Thus, the net force $\vec{F}$ acting on the particle is:

$$\vec{F} = \vec{F}_e + \vec{f}. \tag{3}$$

The electrically charged particle is an electron with charge $q = -e$ moving whithin an electric field directed negatively along the X axis, that is: $\vec{E} = -E\hat{\imath}$, Eq. (1) takes the form:

$$\vec{F}_e = q\vec{E} = (-e)(-E\hat{\imath}) = eE\hat{\imath}\,.$$

Substituting this last expression and Eq. (2) in Eq. (3):

$$\vec{F} = eE\hat{\imath} - \alpha v\hat{\imath}\,.$$

As the movement is one-dimensional, in the direction of the axis, the previous equation can be simplified as:

$$F = eE - \alpha v\,. \tag{4}$$

Eq. (4) will be discussed in the following sections considering both the classical (Section 3.1) and relativistic (Section 3.2) situations. The resulting force F in Eq. (4) for a particle of mass m can be expressed as:

$$F = \frac{d}{dt}p(t), \tag{5}$$

where $p(t)$ is the momentum of the particle. The classical form of the moment $p(t)$ will be treated in Section 3.1 and the relativistic form in Section 3.2.

Results and discussion

Classical formulation

Using the classical form of momentum for a particle:

$$p(t) = mv(t), \tag{6}$$

and substituting in Eq. (5) we have:

$$F = \frac{d}{dt}mv(t) = m\frac{dv(t)}{dt}. \tag{7}$$

Substituting Eq. (7) into Eq. (4) we have:

$$m\frac{dv}{dt} = eE - \alpha v\,. \tag{8}$$

Eq. (8) can be solved exactly as follows:

$$mdv = (eE - \alpha v)dt \Rightarrow \frac{mdv}{eE - \alpha v} = dt \Rightarrow$$

$$\Rightarrow \frac{m\,dv}{\alpha v - eE} = -dt \Rightarrow \frac{mdv}{eE\left(\frac{\alpha v}{eE} - 1\right)} = -dt \Rightarrow \frac{dv}{\left(\frac{\alpha v}{eE} - 1\right)} = -\frac{eE}{m}dt$$

Defining $\Gamma = \alpha/eE$, we have:

$$\frac{dv}{(\Gamma v - 1)} = -\frac{eE}{m}dt \Rightarrow \int_0^v \frac{dv}{(\Gamma v - 1)} = -\int_0^t \frac{eE}{m}dt\,,$$

and using that (Gradshteyn; RYZHIK, 2007):

$$\int \frac{dx}{\Gamma x - 1} = \frac{\ln(\Gamma x - 1)}{\Gamma}$$

we have:

$$\frac{\ln(\Gamma v - 1)}{\Gamma}\bigg|_0^v = -\frac{eE}{m}t\bigg|_0^t \Rightarrow \ln\left(\frac{\Gamma v - 1}{-1}\right) = -\frac{eE\Gamma}{m}t \Rightarrow$$

$$\Rightarrow \ln(1 - \Gamma v) = -\frac{eE\Gamma}{m}t \Rightarrow 1 - \Gamma v = e^{-eE\Gamma t/m} \Rightarrow$$

$$\Rightarrow 1 - e^{-eE\Gamma t/m} = \Gamma v\,.$$

Remembering that $\Gamma = \alpha/eE$ and isolating , we obtain:

$$v(t) = \frac{eE}{\alpha}\left[1 - e^{-\alpha t/m}\right]. \tag{9}$$

Figure 1 shows the velocity as a function of time for an electron using Eq. (9). To obtain numerical results, the following values were used: e = 1.60 x 10^{-19} C, m = 9,11 x 10^{-31} kg, α = $10^{-15\ \mathrm{kg/s}}$ and E = 100 kV/cm. This value of α is a typical value in order of magnitude in the study of carrier transport in semiconductors in an ohmic regime (Rodrigues; Vasconcellos; Luzzi, 2013), and the value of the electric field is common in experimental research involving semiconductors (Rodrigues; Vasconcellos; Luzzi, 2007). From Fig. 1 it can be seen that after a short transient state the velocity reaches a "terminal value". This is a consequence of the fact that when t tending to infinity in Eq. (9) we have: $v_{\mathrm{est}} = eE/\alpha$, where v_{est} is called the "steady-state velocity". Thus, v_{est} is directly proportional to the intensity of the electric field E and inversely proportional to α, that is, the smaller α, the higher v_{est}.

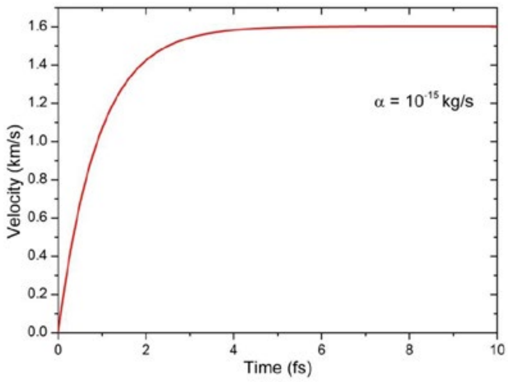

Figure 1. Electron velocity as a function of time. E =100 kV/cm and α = 10^{-15} kg/s
Source: Graphic by Author.

Figure 2 shows the speed of electron as a function of time under the same conditions as Fig. 1, however, adopting α = 0.5 x 10^{-20} kg/s (red line) and α = 0.4 x 10^{-20} kg/s (green line). The dashed line in Fig. 2 indicates the value of the speed of light c. As explained in the previous paragraph, the value of α is decisive for the terminal value of the speed. Fig. 2 clearly shows that Eq. (9) cannot be used in this case, since at a certain instant $v(t)$ becomes greater than the velocity of light c. This extrapolation of the speed of electron greater than c will be eliminated in the next section using relativistic equations. Therefore, the need to use the relativistic form is linked to the value adopted for α.

It is important to note that for the solution to Eq. (8) is simply:

$$v(t) = \frac{eE}{m}t\,, \tag{10}$$

that is, the velocity $v(t)$ increases infinitely with time t, as illustrated by the blue line in Fig. 2. This behavior is also eliminated using relativistic formalism.

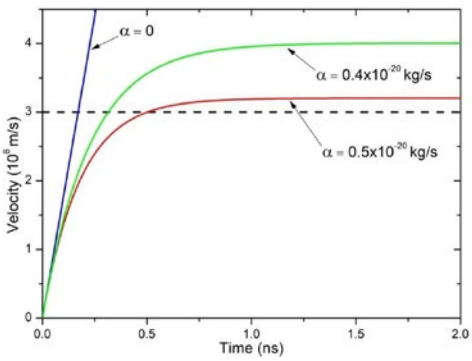

Figure 2. Electron speed as a function of time. E **= 100 kV/cm and** α **= 10^{-21} kg/s**
Source: Graphic by Author.

Relativistic formulation

The relativistic momentum is given by (Resnick, 1968):

$$p = \gamma m_0 v, \tag{11}$$

where m_0 is the rest mass of the particle and γ is the Lorentz factor (Resnick, 1968):

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{12}$$

where is the speed of light. So Eq. (11) becomes:

$$p = \frac{1}{\sqrt{1-v^2/c^2}}m_0 v = \frac{m_0 v}{\sqrt{v^2\left(\frac{1}{v^2}-\frac{1}{c^2}\right)}} = \frac{m_0 v}{v\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}} = \frac{m_0}{\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}}$$

$$p = m_0\left(\frac{1}{v^2}-\frac{1}{c^2}\right)^{-1/2}. \tag{13}$$

To determine the relativistic force we substitute Eq. (12) in Eq. (5):

$$F = \frac{dp}{dt} = \frac{d}{dt} m_0 \left(\frac{1}{v^2} - \frac{1}{c^2}\right)^{-1/2} = -\frac{1}{2}\left(\frac{1}{v^2} - \frac{1}{c^2}\right)^{-3/2} (-2v^{-3}) \frac{dv}{dt} m_0$$

$$F = \frac{v^{-3}}{\left(\frac{1}{v^2} - \frac{1}{c^2}\right)^{3/2}} a m_0 = \frac{v^{-3}\, m_0 a}{\left[\frac{1}{v^2}\left(1 - \frac{v^2}{c^2}\right)\right]^{3/2}}$$

$$F = \frac{v^{-3}\, m_0 a}{\left[\frac{1}{v^3}(1-\beta^2)^{3/2}\right]} = \frac{m_0 a}{\left[(1-\beta^2)^{1/2}\right]^3} = \left[\frac{1}{\sqrt{1-\beta^2}}\right]^3 m_0 a$$

and finally:

$$F = \gamma^3 m_0 \frac{dv(t)}{dt} \,. \tag{14}$$

This is not a new result, being present in standard textbooks on the topic. Substituting Eq. (14) into Eq. (4):

$$\gamma^3 m_0 \dot{v}(t) = qE - \alpha v(t), \tag{15}$$

where the dot over v indicates the derivative of v to concerning time. In Eq. (15) m_0 is the rest mass of the particle and the Lorentz factor γ can be written as:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \,, \tag{16}$$

where β is the velocity parameter:

$$\beta = v/c \,. \tag{17}$$

An attempt to analytically solve Eq. (15) using expansions was made, without success, in Appendix A, resulting in:

$$\sqrt{6}\ \mathrm{ArcTanh}\left(\sqrt{3}\beta/2\right) + 2A\mathrm{Log}(A\beta - 1) - A\mathrm{Log}(3\beta^2 - 2) = \frac{F_e}{2m_0 c} t \,,$$

which is a transcendental equation in $\beta(t)$, that is, it is not possible to isolate the variable $\beta(t)$. Differential equation (15) can only be solved numerically. In this article we use Mathematica software (Wolfram […], 2024). The code is in Appendix B.

After solving Eq. (15) computationally, Figure 3 shows the velocity parameter β as a function of time for an electron considering the same value of the electric field and α used in Figure 2. The dashed line indicates the limiting value for the particle's velocity, that is, $v(t)$ cannot be greater than the speed of light c. It is noted that with the introduction of the relativistic equations, the problem of extrapolation of the speed of light shown in Fig. 2 is then eliminated.

It is worth highlighting the particular situation for $\alpha = 0$, that is when there is no drag force. In the classical formulation, the speed of the particle, subjected to a finite force ($\vec{F}_e = q\vec{E}$), increases infinitely because its mass is constant. However, in the relativistic formulation, the mass of the particle is not a constant; it increases with the speed of the particle, causing an asymptote at the speed of light, already predicted without the drag force.

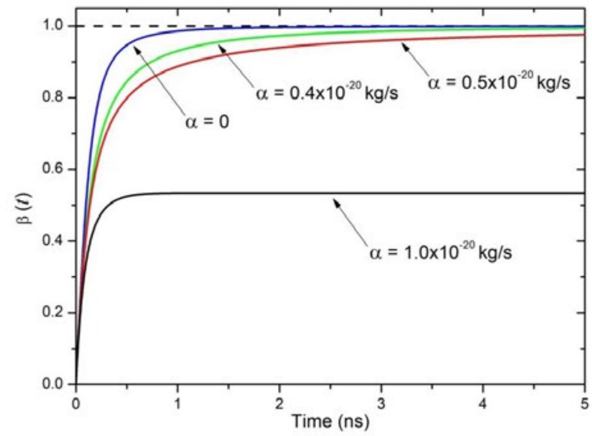

Figure 3. Velocity parameter β as a function of time. E = 100 kV/cm and several values of α.
Source: Graphic by Author.

Final considerations

Given by $\vec{F}_e = q\vec{E}$, a force $\vec{f}$ acts as resistance to the electron movement, which is proportional to the speed: $\vec{f} = -\alpha\vec{v}(t)$, where α is the constant of proportionality. In classical mechanics, it is possible to derive an analytical expression for velocity. However, this classical expression fails to hold true for large values of particle velocity. Relativistic mechanics addresses this issue, but it does not yield an analytical formula for velocity. Instead, in the relativistic case, velocity must be determined using numerical methods through computational techniques.

The main objective of this work is to show the impossibility of determining an analytical expression for the speed of a particle subjected to a finite force and with a resistance force to movement proportional to the speed and we did not find experimental results that are directly related to this work. A possible continuation of this work could be the determination of the radiation emitted by the charged particle accelerated by the electric field, verifying the difference between the situation with and without a force of resistance to movement.

Appendix A

Taylor series expansion of the Lorentz factor in Eq. (15)

Let us try to analytically solve Eq. (15) by expanding the Lorentz factor in a Taylor series in the variable β. Using $v = c\beta$ and $\gamma = 1/\sqrt{1-\beta^2}$, Eq. (15) takes the form:

$$\left(\frac{1}{\sqrt{1-\beta^2}}\right)^3 m_0 c\frac{d\beta}{dt} = F_e - \alpha c\beta$$

$$\frac{1}{(1-\beta^2)^{3/2}} m_0 c\frac{d\beta}{dt} = F_e - \alpha c\beta$$

Considering in this equation (Rodrigues; Vasconcellos; Luzzi, 2017):

$$(1-\beta^2)^{3/2} \approx 1 - \frac{3}{2}\beta^2$$

we have:

$$\frac{m_0 c \frac{d\beta}{dt}}{1-3\beta^2/2} = F_e - \alpha c \beta$$

$$m_0 c \frac{d\beta}{dt} = \left(1 - \frac{3}{2}\beta^2\right)(F_e - \alpha c \beta)$$

$$m_0 c \, d\beta = F_e \left[1 - \frac{\alpha c}{F_e}\beta - \frac{3}{2}\beta^2 + \frac{3\alpha c}{2F_e}\beta^3\right] dt$$

$$\left[1 - \frac{\alpha c}{F_e}\beta - \frac{3}{2}\beta^2 + \frac{3\alpha c}{2F_e}\beta^3\right]^{-1} d\beta = \frac{F_e}{m_0 c} dt$$

$$\int_0^\beta \left[1 - \frac{\alpha c}{F_e}\beta - \frac{3}{2}\beta^2 + \frac{3\alpha c}{2F_e}\beta^3\right]^{-1} d\beta = \int_0^t \frac{F_e}{m_0 c} dt$$

Defining $A = 3\alpha c/2F_e$, the previous equation takes the form:

$$\int_0^\beta \left[1 - A\beta - \frac{3}{2}\beta^2 + \frac{3}{2}A\beta^3\right]^{1} d\beta = \frac{F_e}{m_0 c} t$$

and solving the integral (GRADSHTEYN; RYZHIK, 2007), we have:

$$\sqrt{6}\,\mathrm{ArcTanh}\left(\sqrt{3\beta/2}\right) + 2A\mathrm{Log}(A\beta - 1) - A\mathrm{Log}(3\beta^2 - 2) = \frac{F_e}{2m_0 c} t$$

this being a transcendental equation in , that is, it is not possible to isolate the variable $\beta(t)$.

Appendix B

Code using Mathematica software

```
(*******************************************************************)
(*Differential System Constants in the MKS System*)
(*resistance force proportional to velocity*)
(*Relativistic Form*)

field=1*10^7;
tf=5*10^-9;
m0=9.109*10^-31;
charge=1.602*10^-19;
c=299792458;
alfa=1*10^-21;
beta=v[t]/c;
```

```
gama=1/Sqrt[1-beta^2];
Fe= charge*field;

sol=Evaluate[NDSolve[{
gama^3*m0*v'[t]==Fe-alfa*v[t],v[0]==0},{v[t]},{t,0,tf}]]

Plot[beta /. sol, {t,0,tf}]

Plot[v[t] /. sol, {t,0,tf}]

(*****************************************************************)
```

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